Character sums with Beatty sequences on Burgess-type intervals

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Abstract

We estimate multiplicative character sums taken on the values of a non-homogeneous Beatty sequence $\{\lfloor \alpha n + \beta \rfloor \ : \ n = 1, 2, \ldots\}$, where $\alpha, \beta \in \mathbb{R}$, and α is irrational. Our bounds are nontrivial over the same short intervals for which the classical character sum estimates of Burgess have been established.

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1 Introduction

For fixed $\alpha, \beta \in \mathbb{R}$, the corresponding *non-homogeneous Beatty sequence* is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} = (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}\,,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$ for every $x \in \mathbb{R}$. Beatty sequences arise in a variety of apparently unrelated mathematical settings, and because of their versatility, the arithmetic properties of these sequences have been extensively explored in the literature; see, for example, [1, 6, 15, 16, 19, 22] and the references contained therein.

In this paper, we study character sums of the form

$$S_k(\alpha, \beta, \chi; N) = \sum_{n \leqslant N} \chi(\lfloor \alpha n + \beta \rfloor),$$

where α is *irrational*, and χ is a *non-principal character modulo* k. In the special case that $k = p$ is a prime number, the sums $S_p(\alpha, \beta, \chi; N)$ have been previously studied and estimated nontrivially for $N \geqslant p^{1/3+\varepsilon}$, where $\varepsilon > 0$; see [2, 3].

Here, we show that the approach of [1] (see also [6, 18]), combined with a bound on sums of the form

$$U_k(t, \chi; M_0, M) = \sum_{M_0 < m \leqslant M} \chi(m)\, \mathbf{e}(tm) \qquad (t \in \mathbb{R}), \tag{1}$$

where $\mathbf{e}(x) = \exp(2\pi i x)$ for all $x \in \mathbb{R}$, yields a nontrivial bound on the sums $S_k(\alpha, \beta, \chi; N)$ for all sufficiently large N (see Theorem 4.1 below for a precise statement). In particular, in the case that $k = p$ is prime, we obtain a nontrivial bound for all $N \geqslant p^{1/4+\varepsilon}$, which extends the results found in [2, 3].

It has recently been shown in [5] that for a prime p the least positive quadratic non-residue modulo p among the terms of a Beatty sequence is of size at most $p^{1/(4e^{1/2})+o(1)}$, a result which is complementary to ours. However, the underlying approach of [5] is very different and cannot be used to bound the sums $S_k(\alpha, \beta, \chi; N)$.

We remark that one can obtain similar results to ours by using bounds for double character sums, such as those given in [13]. The approach of this paper, however, which dates back to [1], seems to be more general and can

be used to estimate similar sums with many other arithmetic functions $f(m)$ provided that appropriate upper bounds for the sums

$$V(t, f; M_0, M) = \sum_{M_0 < m \leqslant M} f(m)\, \mathbf{e}(tm) \qquad (t \in \mathbb{R})$$

are known. Such estimates have been obtained for the characteristic functions of primes and of smooth numbers (see [11] and [12], respectively), as well as for many other functions. Thus, in principle one can obtain asymptotic formulas for the number of primes or smooth numbers in a segment of a Beatty sequence (in the case of smooth numbers, this has been done in [4] by a different method).

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2 Notation

Throughout the paper, the implied constants in the symbols O and $\ll$ may depend on α and ε but are absolute otherwise. We recall that the notations $U = O(V)$ and $U \ll V$ are equivalent to the assertion that the inequality $|U| \leqslant c\,V$ holds for some constant $c > 0$.

We also use the symbol $o(1)$ to denote a function that tends to 0 and depends only on α and ε. It is important to note that our bounds are uniform with respect all of the involved parameters other than α and ε; in particular, our bounds are uniform with respect to β. In particular, the latter means that our bounds also apply to the shifted sums of the form

$$\sum_{M+1 \leqslant n \leqslant M+N} \chi(\lfloor \alpha n + \beta \rfloor) = \sum_{n \leqslant N} \chi(\lfloor \alpha n + \alpha M + \beta \rfloor),$$

and these bounds are uniform for all integers M.

In what follows, the letters m and n always denote non-negative integers unless indicated otherwise.

We use $\lfloor x \rfloor$ and $\{x\}$ to denote the greatest integer $\leqslant x$ and the fractional part of x, respectively.

Finally, recall that the *discrepancy* $D(M)$ of a sequence of (not necessarily distinct) real numbers $a_1, \ldots, a_M \in [0,1)$ is defined by

$$D(M) = \sup_{\mathcal{I} \subseteq [0,1)} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|, \tag{2}$$

where the supremum is taken all subintervals $\mathcal{I} = (c,d)$ of the interval $[0,1)$, $V(\mathcal{I}, M)$ is the number of positive integers $m \leqslant M$ such that $a_m \in \mathcal{I}$, and $|\mathcal{I}| = d - c$ is the length of $\mathcal{I}$.

3 Preliminaries

It is well known that for every irrational number α, the sequence of fractional parts $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots,$ is *uniformly distributed modulo* 1 (for instance, see [17, Example 2.1, Chapter 1]). More precisely, let $D_{\alpha,\beta}(M)$ denote the discrepancy of the sequence $(a_m)_{m=1}^{M}$, where

$$a_m = \{\alpha m + \beta\} \qquad (m = 1, 2, \ldots, M).$$

Then, we have:

Lemma 3.1. *Let α be a fixed irrational number. Then, for all $\beta \in \mathbb{R}$ we have*

$$D_{\alpha,\beta}(M) \leqslant 2D_{\alpha,0}(M) = o(1) \qquad (M \to \infty),$$

where the function implied by $o(1)$ depends only on α.

When more information about α is available, the bound of Lemma 3.1 can be made more explicit. For this, we need to recall some familiar notions from the theory of *Diophantine approximations*.

For an irrational number α, we define its *type* τ by the relation

$$\tau = \sup\Big\{\vartheta \in \mathbb{R} \ : \ \liminf_{q \to \infty,\ q \in \mathbb{Z}^+} q^{\vartheta} \, \|\alpha q\| = 0\Big\}.$$

Using *Dirichlet's approximation theorem*, it is easy to see that $\tau \geqslant 1$ for every irrational number α. The celebrated theorems of Khinchin [14] and of Roth [20] assert that $\tau = 1$ for almost all real numbers α (with respect to Lebesgue measure) and all algebraic irrational numbers α, respectively; see also [7, 21].

The following result is taken from [17, Theorem 3.2, Chapter 2]:

Lemma 3.2. *Let α be a fixed irrational number of type $\tau < \infty$. Then, for all $\beta \in \mathbb{R}$ we have*

$$D_{\alpha,\beta}(M) \leqslant M^{-1/\tau+o(1)} \qquad (M \to \infty),$$

where the function implied by $o(1)$ depends only on α.

Next, we record the following property of type:

Lemma 3.3. *If α is an irrational number of type $\tau < \infty$ then so are α^{-1} and $a\alpha$ for any integer $a \geqslant 1$.*

Finally, we need the following elementary result, which describes the set of values taken by the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ in the case that $\alpha > 1$:

Lemma 3.4. *Let $\alpha > 1$. An integer m has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer n if and only if*

$$0 < \{\alpha^{-1}(m - \beta + 1)\} \leqslant \alpha^{-1}.$$

The value of n is determined uniquely by m.

Proof. It is easy to see that an integer m has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer n if and only if the inequalities

$$\frac{m-\beta}{\alpha} \leqslant n < \frac{m-\beta+1}{\alpha}$$

hold, and since $\alpha > 1$ the value of n is determined uniquely. □

4 Character Sums

For every real number $\varepsilon > 0$ and integer $k \geqslant 1$, we put

$$B_\varepsilon(k) = \begin{cases} k^{1/4+\varepsilon} & \text{if } k \text{ is prime;} \\ k^{1/3+\varepsilon} & \text{if } k \text{ is a prime power;} \\ k^{3/8+\varepsilon} & \text{otherwise.} \end{cases} \qquad (3)$$

Theorem 4.1. *Let $\alpha > 0$ be a fixed irrational number, and let $\varepsilon > 0$ be fixed. Then, uniformly for all $\beta \in \mathbb{R}$, all non-principal multiplicative characters χ modulo k, and all integers $N \geqslant B_\varepsilon(k)$, we have*

$$S_k(\alpha, \beta, \chi; N) = o(N) \qquad (k \to \infty),$$

where the function implied by $o(N)$ depends only on α and ε.

Proof. We can assume that $\varepsilon < 1/10$, and this implies that $B_\varepsilon(k) \leqslant k^{2/5}$ in all cases. Observe that it suffices to prove the result in the case that $B_\varepsilon(k) \leqslant N \leqslant k^{1/2}$. Indeed, assuming this has been done, for any $N > k^{1/2}$ we put $N_0 = \lfloor k^{9/20} \rfloor$ and $t = \lfloor N/N_0 \rfloor$; then, since $B_\varepsilon(k) \leqslant N_0 \leqslant k^{1/2}$ we have

$$\begin{aligned} S_k(\alpha,\beta,\chi;N) &= \sum_{j=0}^{t-1} \sum_{n \leqslant N_0} \chi(\lfloor \alpha(n+jN_0)+\beta \rfloor) + \sum_{tN_0 < n \leqslant N} \chi(\lfloor \alpha n + \beta \rfloor) \\ &= \sum_{j=0}^{t-1} S_k(\alpha, \beta + \alpha j N_0, \chi; N_0) + O(N_0) \\ &= o(tN_0) + O\big(Nk^{-1/20}\big) = o(N) \qquad (k \to \infty) \end{aligned}$$

using the fact that our bounds are uniform with respect to β.

We first treat the case that $\alpha > 1$. Put $\gamma = \alpha^{-1}$, $\delta = \alpha^{-1}(1-\beta)$, $M_0 = \lfloor \alpha + \beta - 1 \rfloor$, and $M = \lfloor \alpha N + \beta \rfloor$. From Lemma 3.4 we see that

$$S_k(\alpha,\beta,\chi;N) = \sum_{\substack{M_0 < m \leqslant M \\ 0 < \{\gamma m + \delta\} \leqslant \gamma}} \chi(m) = \sum_{M_0 < m \leqslant M} \chi(m)\, \psi(\gamma m + \delta), \qquad (4)$$

where $\psi(x)$ is the periodic function with period one for which

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \leqslant \gamma; \\ 0 & \text{if } \gamma < x \leqslant 1. \end{cases}$$

By a classical result of Vinogradov (see [23, Chapter 2, Lemma 2]) it is known that for any Δ such that

$$0 < \Delta < \frac{1}{8} \qquad \text{and} \qquad \Delta \leqslant \frac{1}{2} \min\{\gamma, 1-\gamma\},$$

there is a real-valued function $\psi_\Delta(x)$ with the following properties:

- $\psi_\Delta(x)$ is periodic with period one;
- $0 \leqslant \psi_\Delta(x) \leqslant 1$ for all $x \in \mathbb{R}$;
- $\psi_\Delta(x) = \psi(x)$ if $\Delta \leqslant x \leqslant \gamma - \Delta$ or $\gamma + \Delta \leqslant x \leqslant 1 - \Delta$;

- $\psi_\Delta(x)$ can be represented as a Fourier series

$$\psi_\Delta(x) = \gamma + \sum_{j=1}^{\infty} \left(g_j \, \mathbf{e}(jx) + h_j \, \mathbf{e}(-jx) \right),$$

  where the coefficients g_j, h_j satisfy the uniform bound

$$\max\{|g_j|, |h_j|\} \ll \min\{j^{-1}, j^{-2}\Delta^{-1}\} \qquad (j \geqslant 1).$$

Therefore, from (4) we derive that

$$S_k(\alpha, \beta, \chi; N) = \sum_{M_0 < m \leqslant M} \chi(m)\psi_\Delta(\gamma m + \delta) + O(V(\mathcal{I}, M_0, M)), \qquad (5)$$

where $V(\mathcal{I}, M_0, M)$ denotes the number of integers $M_0 < m \leqslant M$ such that

$$\{\gamma m + \delta\} \in \mathcal{I} = [0, \Delta) \cup (\gamma - \Delta, \gamma + \Delta) \cup (1 - \Delta, 1).$$

Since $|\mathcal{I}| \ll \Delta$, it follows from Lemma 3.1 and the definition (2) that

$$V(\mathcal{I}, M_0, M) \ll \Delta N + o(N), \qquad (6)$$

where the implied function $o(N)$ depends only on α.

To estimate the sum in (5), we insert the Fourier expansion for $\psi_\Delta(\gamma m+\delta)$ and change the order of summation, obtaining

$$\sum_{M_0 < m \leqslant M} \chi(m)\psi_\Delta(\gamma m + \delta) = \gamma \, U_k(0, \chi; M_0, M)$$
$$+ \sum_{j=1}^{\infty} g_j \, \mathbf{e}(\delta j) \, U_k(\gamma j, \chi; M_0, M) + \sum_{j=1}^{\infty} h_j \, \mathbf{e}(-\delta j) \, U_k(-\gamma j, \chi; M_0, M),$$

where the sums $U_k(t, \chi; M_0, M)$ are defined by (1).

Since $M - M_0 \ll N$, using the well known results of Burgess [8, 9, 10] on bounds for partial Gauss sums, it follows that for any fixed $\varepsilon > 0$ there exists $\eta > 0$ such that

$$U_k(a/k, \chi; M_0, M) \ll N^{1-\eta} \qquad (7)$$

holds uniformly for all $N \geqslant B_\varepsilon(k)$ and all integers a; clearly, we can assume that $\eta \leqslant 1/10$.

Put $r = \lfloor \gamma k \rfloor$. Then, for any integer n, we have

$$\mathbf{e}(\gamma n) - \mathbf{e}(rn/k) \ll |\gamma n - rn/k| \leqslant |n|k^{-1},$$

which implies that

$$U_k(\gamma j, \chi; M_0, M) = U_k(rj/k, \chi; M_0, M) + O(N^2 k^{-1} |j|).$$

Using (7) in the case that $|j| \leqslant kN^{-1-\eta}$ we derive that

$$U_k(\gamma j, \chi; M_0, M) \ll N^{1-\eta},$$

and for $|j| > kN^{-1-\eta}$ we use the trivial bound

$$\big| U_k(\gamma j, \chi; M_0, M) \big| \ll N.$$

Consequently,

$$\begin{aligned}
&\sum_{M_0 < m \leqslant M} \chi(m) \psi_\Delta(\gamma m + \delta) \\
&\qquad \ll N^{1-\eta} \sum_{j \leqslant kN^{-1-\eta}} (|g_j| + |h_j|) + N \sum_{j > kN^{-1-\eta}} (|g_j| + |h_j|) \\
&\qquad \ll N^{1-\eta} \sum_{j \leqslant kN^{-1-\eta}} j^{-1} + N\Delta^{-1} \sum_{j > kN^{-1-\eta}} j^{-2} \\
&\qquad \ll N^{1-\eta} \log k + N^{2+\eta} \Delta^{-1} k^{-1}.
\end{aligned}$$

Since $N^2 \leqslant k \leqslant N^4$, we see that

$$\sum_{M_0 < m \leqslant M} \chi(m) \psi_\Delta(\gamma m - \delta) \ll N^{1-\eta} \log N + N^\eta \Delta^{-1}. \tag{8}$$

Inserting the bounds (6) and (8) into (5), choosing $\Delta = N^{(\eta-1)/2}$, and taking into account that $0 < \eta \leqslant 1/10$, we complete the proof in the case that $\alpha > 1$.

If $\alpha < 1$, put $a = \lceil \alpha^{-1} \rceil$ and write

$$\begin{aligned}
S_k(\alpha, \beta, \chi; N) &= \sum_{n \leqslant N} \chi(\lfloor \alpha n + \beta \rfloor) \\
&= \sum_{j=0}^{a-1} \sum_{m \leqslant (N-j)/a} \chi(\lfloor \alpha a m + \alpha j + \beta \rfloor) \\
&= \sum_{j=0}^{a-1} S_k(\alpha a, \alpha j + \beta, \chi; (N-j)/a).
\end{aligned}$$

Applying the preceding argument with the irrational number $\alpha a > 1$, we conclude the proof. $\square$

For an irrational number α of type $\tau < \infty$, we proceed as in the proof of Theorem 4.1, using Lemma 3.2 instead of Lemma 3.1, and also applying Lemma 3.3; this yields the following statement:

Theorem 4.2. *Let $\alpha > 0$ be a fixed irrational number of type $\tau < \infty$. For every fixed $\varepsilon > 0$ there exists $\rho > 0$, which depends only on ε and τ, such that for all $\beta \in \mathbb{R}$, all non-principal multiplicative characters χ modulo k, and all integers $N \geqslant B_\varepsilon(k)$, we have*

$$S_k(\alpha, \beta, \chi; N) \ll N k^{-\rho}.$$

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